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The spectrum of a locally coherent category

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Abstract

A topology on the spectrum of a locally coherent Grothendieck category is introduced. The closed subsets are related to certain localizing subcategories which are characterized in terms of Serre subcategories of the full subcategory of finitely presented objects.

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0. Introduction

A Grothendieck category $\mathscr{A}$ is said to be locally coherent provided that $\mathscr{A}$ has a generating set of finitely presented objects and the full subcategory $\mathrm{fp}(\mathscr{A})$ of finitely presented objects in $\mathscr{A}$ is abelian. The spectrum $\mathrm{sp}(\mathscr{A})$ of $\mathscr{A}$ is the set of isomorphism classes of indecomposable injective objects in $\mathscr{A}$. We show that this set carries a natural topology and it is the purpose of this paper to establish a natural and bijective correspondence between the following structures which arise for each locally coherent category $\mathscr{A}$:

- Serre subcategories of $\mathrm{fp}(\mathscr{A})$,
- hereditary torsion theories of finite type for $\mathscr{A}$,
- closed subsets of $\mathrm{sp}(\mathscr{A})$.

This analysis is motivated by some construction which reduces the theory of purity for a locally finitely presented category with products to the study of injectives in a locally coherent category (e.g. [3]). Model theory of modules is a classical example of this theory. For instance, given a ring Λ the category $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ of additive

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functors from the category of finitely presented Λ^{op}-modules to the category of abelian groups is locally coherent, and the functor

$$\mathrm{Mod}(\Lambda) \to (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}), \quad M \mapsto M \otimes_\Lambda -$$

identifies the pure-injective Λ-modules with the injective objects of $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$.

Central parts of this paper were presented at a workshop in Bielefeld in November 1993 which was devoted to connections between model theory and representation theory of finite dimensional algebras. I wish to thank all participants for exciting discussions and encouragement. In particular I would like to mention I. Herzog and M. Prest. Finally, I should refer to a recent paper of I. Herzog [5] which also discusses locally coherent categories.

1. Locally finitely presented abelian categories

We recall some terminology and some well-known facts about locally finitely presented categories and Grothendieck categories. Let $\mathscr{A}$ be an additive category with direct limits. An object $X \in \mathscr{A}$ is *finitely presented* provided that $\mathrm{Hom}(X, \ )$ commutes with direct limits and we denote by $\mathrm{fp}(\mathscr{A})$ the full subcategory of finitely presented objects in $\mathscr{A}$. The category $\mathscr{A}$ is said to be *locally finitely presented* if the isomorphism classes of $\mathrm{fp}(\mathscr{A})$ form a set and every object in $\mathscr{A}$ is a direct limit of objects in $\mathrm{fp}(\mathscr{A})$. An abelian category $\mathscr{A}$ is locally finitely presented if and only if it is a Grothendieck category with a generating set of finitely presented objects [1, Satz 1.5; 2, 2.4].

Fix now a locally finitely presented abelian category $\mathscr{A}$. An object in $\mathscr{A}$ is *finitely generated* if it is a quotient of some finitely presented object and it is easily seen that any object in $\mathscr{A}$ is a direct limit of finitely generated subobjects. In particular, one has the following lemma.

Lemma 1.1. *An object $X \in \mathscr{A}$ is finitely generated iff for any epimorphism $\varphi : Y \to X$ there is a finitely generated subobject U of Y such that $\varphi(U) = X$.*

Proof. Straightforward. □

The category $\mathscr{A}$ is completely determined by $\mathrm{fp}(\mathscr{A})$. In fact, $\mathrm{fp}(\mathscr{A})$ is additive, has cokernels and the functor

$$\mathscr{A} \to \mathrm{Lex}(\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathrm{Hom}(\ , X)|_{\mathrm{fp}(\mathscr{A})}$$

from $\mathscr{A}$ into the category of additive left exact functors from $\mathrm{fp}(\mathscr{A})^{\mathrm{op}}$ to Ab is an equivalence [1, Satz 2.4]. Given a full additive subcategory $\mathscr{C}$ of $\mathrm{fp}(\mathscr{A})$ we denote by $\vec{\mathscr{C}}$ the full subcategory of $\mathscr{A}$ which consists of direct limits $\varinjlim X_i$ with $X_i \in \mathscr{C}$ for all i. Note that this subcategory is closed under direct limits in $\mathscr{A}$ and that the finitely presented objects in $\vec{\mathscr{C}}$ are precisely the direct summands of objects in $\mathscr{C}$ [2, Theorem 4.1].

Finally, we recall that $\mathscr{A}$ is *locally coherent* provided that finitely generated subobjects of finitely presented objects are finitely presented, equivalently if $\text{fp}(\mathscr{A})$ is abelian [5, Proposition 2.2].

2. Torsion theories and localization

In this section we discuss certain localizing subcategories which arise naturally for any locally coherent category. We begin with some definitions. Let $\mathscr{A}$ be an abelian category. A pair $(\mathscr{T},\mathscr{F})$ of full subcategories of $\mathscr{A}$ is said to be a *torsion theory* for $\mathscr{A}$ provided that

(1) $\text{Hom}(\mathscr{T},\mathscr{F}) = 0$;

(2) $\text{Hom}(\mathscr{T},X) = 0$ implies $X \in \mathscr{F}$ for all $X \in \mathscr{A}$;

(3) $\text{Hom}(X,\mathscr{F}) = 0$ implies $X \in \mathscr{T}$ for all $X \in \mathscr{A}$;

(4) for all $X \in \mathscr{A}$ there exists a subobject $Y \subseteq X$ such that $Y \in \mathscr{T}$ and $X/Y \in \mathscr{F}$.

We denote by $t: \mathscr{A} \to \mathscr{T}$ the functor which assigns to $X \in \mathscr{A}$ the largest subobject $t(X)$ of X belonging to $\mathscr{T}$. A torsion theory $(\mathscr{T},\mathscr{F})$ is called *hereditary* if $\mathscr{T}$ is closed under subobjects, equivalently if t is left exact. A full subcategory $\mathscr{C}$ of $\mathscr{A}$ is called a *Serre subcategory* provided that for every exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ the object Y is in $\mathscr{C}$ iff both X and Z are in $\mathscr{C}$.

We fix a Serre subcategory $\mathscr{C}$ of $\mathscr{A}$. The *quotient category* $\mathscr{A}/\mathscr{C}$ of $\mathscr{A}$ relative to $\mathscr{C}$ is defined as follows. The objects of $\mathscr{A}/\mathscr{C}$ are those of $\mathscr{A}$ and $\text{Hom}_{\mathscr{A}/\mathscr{C}}(X,Y) = \varinjlim \text{Hom}_{\mathscr{A}}(X',Y/Y')$ with $X' \subseteq X$, $Y' \subseteq Y$ and $X/X', Y' \in \mathscr{C}$. Again $\mathscr{A}/\mathscr{C}$ is abelian and the canonical *quotient functor* $q: \mathscr{A} \to \mathscr{A}/\mathscr{C}$ with $q(X) = X$ is exact. A Serre subcategory $\mathscr{C}$ is called *localizing* provided that q admits a right adjoint $s: \mathscr{A}/\mathscr{C} \to \mathscr{A}$ which is called *section functor*. The adjointness gives a natural morphism $\xi_X: X \to s\circ q(X)$ for each $X \in \mathscr{A}$ with $\text{Ker}(\xi_X), \text{Coker}(\xi_X) \in \mathscr{C}$, and $\text{Ker}(\xi_X)$ is the maximal subobject of X which belongs to $\mathscr{C}$. An object $X \in \mathscr{A}$ is said to be *$\mathscr{C}$-closed* (*$\mathscr{C}$-torsionfree*) provided that ξ_X is an isomorphism (a monomorphism). Thus the section functor induces an equivalence between $\mathscr{A}/\mathscr{C}$ and the full subcategory of $\mathscr{C}$-closed objects in $\mathscr{A}$. The following lemma collects some well-known facts about the connection between localizing subcategories and torsion theories.

Lemma 2.1. *Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. Consider the following conditions:*

(1) *$(\mathscr{C}, \{X \in \mathscr{A} \mid \text{Hom}(\mathscr{C},X) = 0\})$ is a hereditary torsion theory.*

(2) *The inclusion $\mathscr{C} \to \mathscr{A}$ admits a right adjoint $t: \mathscr{A} \to \mathscr{C}$.*

(3) *$\mathscr{C}$ is a localizing subcategory.*

Then (1) *and* (2) *are equivalent and* (3) *implies* (2). *If $\mathscr{A}$ has injective envelopes, then also* (2) *implies* (3).

Proof. (1) $\Rightarrow$ (2): Take for $t: \mathscr{A} \to \mathscr{C}$ the functor which assigns to $X \in \mathscr{A}$ the largest subobject $t(X)$ of X belonging to $\mathscr{C}$.

(2) $\Rightarrow$ (1): The right adjoint $t: \mathscr{A} \to \mathscr{C}$ induces a morphism $\xi_X: t(X) \to X$ for every $X \in \mathscr{A}$ which is a monomorphism since $\mathscr{C}$ is closed under quotients. It is easily

checked that $\operatorname{Hom}(\mathscr{C}, \operatorname{Coker}(\xi_X)) = 0$, and it follows that $(\mathscr{C}, \{X \in \mathscr{A} \mid \operatorname{Hom}(\mathscr{C}, X) = 0\})$ is a hereditary torsion theory.

(2) ⇒ (3) follows from [3, III.3, Corollaire 1].

(3) ⇒ (2): Let s be a right adjoint of the quotient functor $q: \mathscr{A} \to \mathscr{A}/\mathscr{C}$. Take for $t: \mathscr{A} \to \mathscr{C}$ the functor which assigns to $X \in \mathscr{A}$ the kernel of the canonical morphism $\xi_X : X \to s \circ q(X)$. □

Given a collection $\mathscr{C}$ of objects in an abelian category $\mathscr{A}$, recall that the *right perpendicular category* $\mathscr{C}^{\perp}$ of $\mathscr{C}$ is the full subcategory of all objects $M \in \mathscr{A}$ satisfying $\operatorname{Hom}(X, M) = 0$ and $\operatorname{Ext}^1(X, M) = 0$ for all $X \in \mathscr{C}$.

Lemma 2.2. *If $\mathscr{C}$ is a localizing subcategory, then the section functor induces an equivalence between $\mathscr{A}/\mathscr{C}$ and $\mathscr{C}^{\perp}$.*

Proof. The right perpendicular category $\mathscr{C}^{\perp}$ coincides with the full subcategory of $\mathscr{C}$-closed objects since a $\mathscr{C}$-torsionfree object $X \in \mathscr{A}$ is $\mathscr{C}$-closed iff $\operatorname{Ext}^1(\mathscr{C}, X) = 0$ [3, III.2, Lemme 1]. □

Let $\mathscr{A}$ be an abelian category and $(\mathscr{T}, \mathscr{F})$ a torsion theory for $\mathscr{A}$. If $\mathscr{A}$ has direct limits, then the isomorphism $\operatorname{Hom}(\varinjlim X_i, Y) \cong \varprojlim \operatorname{Hom}(X_i, Y)$ shows that $\mathscr{T}$ is closed under direct limits in $\mathscr{A}$. A torsion theory is said to be of *finite type* provided that the corresponding right adjoint t of the inclusion $\mathscr{T} \to \mathscr{A}$ commutes with direct limits. If direct limits in $\mathscr{A}$ are exact, then $(\mathscr{T}, \mathscr{F})$ is of finite type iff $\mathscr{F}$ is closed under direct limits.

Lemma 2.3. *Let $\mathscr{A}$ be a locally coherent category. Suppose that the pair $(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory of finite type for $\mathscr{A}$. Then $\mathscr{S} = \mathscr{T} \cap \operatorname{fp}(\mathscr{A})$ is a Serre subcategory of $\operatorname{fp}(\mathscr{A})$ with $\mathscr{T} = \vec{\mathscr{S}}$ and $\mathscr{F} = \{X \in \mathscr{A} \mid \operatorname{Hom}(\mathscr{S}, X) = 0\}$.*

Proof. We have $\vec{\mathscr{S}} \subseteq \mathscr{T}$ since $\mathscr{T}$ is closed under direct limits. Now let $X \in \mathscr{T}$ and write $X = \varinjlim X_i$ as direct limit of finitely presented objects. We have $t(Y) \in \vec{\mathscr{S}}$ for all $Y \in \operatorname{fp}(\mathscr{A})$ since $t(Y)$ can be written as direct limit of finitely generated subobjects which belong to $\mathscr{S}$. Thus $X = t(X) \cong \varinjlim t(X_i) \in \vec{\mathscr{S}}$ since $\vec{\mathscr{S}}$ is closed under direct limits and the equality $\mathscr{T} = \vec{\mathscr{S}}$ is shown. It is clear that $\mathscr{F} \subseteq \{X \in \mathscr{A} \mid \operatorname{Hom}(\mathscr{S}, X) = 0\}$. On the other hand, $\operatorname{Hom}(\mathscr{S}, X) = 0$ implies $\operatorname{Hom}(\vec{\mathscr{S}}, X) = 0$ and therefore $X \in \mathscr{F}$ since $\mathscr{T} = \vec{\mathscr{S}}$. This completes the proof. □

Lemma 2.4. *Let $\mathscr{A}$ be an abelian category with exact direct limits and suppose that $\mathscr{C}$ is a localizing subcategory of $\mathscr{A}$. Then $\mathscr{C}$ and $\mathscr{A}/\mathscr{C}$ have direct limits. Consider the following conditions:*

(1) *t commutes with direct limits.*

(2) *s commutes with direct limits.*

Then (2) *implies* (1) *and if $\mathscr{A}$ is locally coherent, then also* (1) *implies* (2).

Proof. Direct limits exist in $\mathscr{C}$ since $\mathscr{C}$ is closed under direct limits taken in $\mathscr{A}$. For a direct limit $\varinjlim X_i$ in $\mathscr{A}/\mathscr{C}$ take $q(\varinjlim s(X_i))$.

(2) $\Rightarrow$ (1): We use the natural exact sequence $0 \to t(X) \to X \xrightarrow{\xi_X} s \circ q(X)$ which is given for each $X \in \mathscr{A}$. By assumption s commutes with direct limits and therefore also $s \circ q$ since q is a left adjoint. From the exactness of direct limits in $\mathscr{A}$ we obtain therefore an isomorphism $\varinjlim t(X_i) \cong t(\varinjlim X_i)$ for each direct limit $\varinjlim X_i$ in $\mathscr{A}$.

(1) $\Rightarrow$ (2): Using the adjointness we have for a direct limit $\varinjlim X_i$ in $\mathscr{A}/\mathscr{C}$ the isomorphism $\varinjlim X_i \cong \varinjlim q \circ s(X_i) \cong q(\varinjlim s(X_i))$. Therefore it suffices to show that $\varinjlim s(X_i)$ is $\mathscr{C}$-closed. We have $t(\varinjlim s(X_i)) \cong \varinjlim t \circ s(X_i) = 0$ since t commutes with direct limits. Thus $\xi_{\varinjlim s(X_i)}$ induces an exact sequence $0 \to \varinjlim s(X_i) \to s \circ q(\varinjlim s(X_i)) \to C \to 0$ for some $C \in \mathscr{C}$. We need to show that $C = 0$. Therefore assume $C \neq 0$. We can choose a non-zero morphism $Z \to C$ for some $Z \in \mathscr{S} = \mathscr{C} \cap \operatorname{fp}(\mathscr{A})$ since $\mathscr{C} = \overrightarrow{\mathscr{S}}$ by Lemma 2.3. This morphism induces an exact sequence $0 \to \varinjlim s(X_i) \to Y \to Z \to 0$ and we obtain from Lemma 1.1 a morphism $V \to Y$ with $V \in \operatorname{fp}(\mathscr{A})$ which gives the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & Z & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \varinjlim s(X_i) & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varinjlim s(X_i) & \longrightarrow & s \circ q(\varinjlim s(X_i)) & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

The object U is finitely presented since $\operatorname{fp}(\mathscr{A})$ is abelian and therefore $U \to \varinjlim s(X_i)$ factorizes as $U \xrightarrow{\varphi} s(X_j) \to \varinjlim s(X_i)$ for some j. The morphism φ induces an element in $\operatorname{Ext}^1(Z, s(X_j))$ which is zero since $\operatorname{Ext}^1(\mathscr{C}, X) = 0$ for all $\mathscr{C}$-closed objects $X \in \mathscr{A}$ [3, III.2, Lemme 1]. Thus $0 \to \varinjlim s(X_i) \to Y \to Z \to 0$ splits and therefore the morphism $Z \to C$ is zero since $\operatorname{Hom}(Z, s \circ q(\varinjlim s(X_i))) = 0$. This contradiction finishes the proof. $\square$

Lemma 2.5. *Let $f : \mathscr{A} \to \mathscr{B}$ be a functor between categories with direct limits. Suppose there exists a right adjoint g which commutes with direct limits. Then $f(X)$ is finitely presented, if X is a finitely presented object in $\mathscr{A}$.*

Proof. Let $X \in \mathscr{A}$ and $\varinjlim Y_i \in \mathscr{B}$. We have the following sequence of morphisms

$$\begin{aligned}
&\varinjlim \operatorname{Hom}(f(X), Y_i) \cong \varinjlim \operatorname{Hom}(X, g(Y_i)) \\
&\quad \xrightarrow{\varphi} \operatorname{Hom}(X, \varinjlim g(Y_i)) \cong \operatorname{Hom}(X, g(\varinjlim Y_i)) \cong \operatorname{Hom}(f(X), \varinjlim Y_i)
\end{aligned}$$

since f and g is a pair of adjoint functors and g commutes with direct limits. Now the assertion follows since φ is an isomorphism if X is finitely presented. $\square$

We say that a localizing subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ with direct limits is of *finite type* if the right adjoint $t : \mathscr{A} \to \mathscr{C}$ of the inclusion commutes with direct limits.

The next result summarizes our discusssion and it shows that this type of localizing subcategories is most natural for locally coherent categories.

Theorem 2.6. *Let $\mathscr{A}$ be a locally coherent category and suppose that $\mathscr{C}$ is a Serre subcategory of $\mathscr{A}$. Then the following are equivalent:*

(1) The inclusion functor $\mathscr{C} \to \mathscr{A}$ admits a right adjoint $t: \mathscr{A} \to \mathscr{C}$.

(2) The quotient functor $q: \mathscr{A} \to \mathscr{A}/\mathscr{C}$ admits a right adjoint $s: \mathscr{A}/\mathscr{C} \to \mathscr{A}$.

If the above conditions (1) and (2) are satisfied, then the following are equivalent:

(3) t commutes with direct limits.

(4) s commutes with direct limits.

If the above conditions (1)–(4) are satisfied, then $\mathscr{C}$ and $\mathscr{A}/\mathscr{C}$ are locally coherent and the following diagram of functors commutes where the unlabeled functors are inclusions:

$$\begin{array}{ccccc} \mathrm{fp}(\mathscr{C}) & \longrightarrow & \mathrm{fp}(\mathscr{A}) & \xrightarrow{\mathrm{fp}(q)} & \mathrm{fp}(\mathscr{A}/\mathscr{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{C} & \longrightarrow & \mathscr{A} & \xrightarrow{q} & \mathscr{A}/\mathscr{C} \end{array}$$

Moreover, there is a unique functor $f: \mathrm{fp}(\mathscr{A})/\mathrm{fp}(\mathscr{C}) \to \mathrm{fp}(\mathscr{A}/\mathscr{C})$ such that $\mathrm{fp}(q) = f \circ p$, where $p: \mathrm{fp}(\mathscr{A}) \to \mathrm{fp}(\mathscr{A})/\mathrm{fp}(\mathscr{C})$ denotes the quotient functor. The functor f is an equivalence.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Lemma 2.1 and (3) $\Leftrightarrow$ (4) from Lemma 2.4. Now suppose (1)–(4). It follows from Lemma 2.1 and Lemma 2.3 that $\mathscr{C}$ is locally coherent. To show that $\mathscr{A}/\mathscr{C}$ is locally coherent first observe that $\mathscr{A}/\mathscr{C}$ has direct limits by Lemma 2.4. Moreover, q sends finitely presented objects to finitely presented objects by Lemma 2.5 since s is a right adjoint of q commuting with direct limits. Furthermore, any object X in $\mathscr{A}/\mathscr{C}$ can be written as a direct limit $X = q(X) = \varinjlim q(X_i)$ of objects in $q(\mathrm{fp}(\mathscr{A}))$. Thus $\mathscr{A}/\mathscr{C}$ is locally finitely presented and the commutativity of the diagram is also shown. It remains to check that there exists a canonical equivalence between $\mathrm{fp}(\mathscr{A})/\mathrm{fp}(\mathscr{C})$ and $\mathrm{fp}(\mathscr{A}/\mathscr{C})$. In particular this would imply that $\mathscr{A}/\mathscr{C}$ is locally coherent since $\mathrm{fp}(\mathscr{A})/\mathrm{fp}(\mathscr{C})$ is abelian. The existence of a unique functor $f: \mathrm{fp}(\mathscr{A})/\mathrm{fp}(\mathscr{C}) \to \mathrm{fp}(\mathscr{A}/\mathscr{C})$ such that $\mathrm{fp}(q) = f \circ p$ follows from the universal property of the quotient functor $p: \mathrm{fp}(\mathscr{A}) \to \mathrm{fp}(\mathscr{A})/\mathrm{fp}(\mathscr{C})$ [3, III.1, Corollaire 2] since $\mathrm{fp}(q)(X) = 0$ for all $X \in \mathrm{fp}(\mathscr{C})$. In fact f is induced from the canonical morphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{fp}(\mathscr{A})/\mathrm{fp}(\mathscr{C})}(X, Y) &= \varinjlim \mathrm{Hom}_{\mathrm{fp}(\mathscr{A})}(X'', Y/Y'') \\ &\to \varinjlim \mathrm{Hom}_{\mathscr{A}}(X', Y/Y') = \mathrm{Hom}_{\mathscr{A}/\mathscr{C}}(X, Y) \end{aligned}$$

for objects $X, Y \in \mathrm{fp}(\mathscr{A})/\mathrm{fp}(\mathscr{C})$ with $X', X'' \subseteq X$, $Y', Y'' \subseteq Y$, $X/X'', Y'' \in \mathrm{fp}(\mathscr{C})$ and $X/X', Y' \in \mathscr{C}$. This is an isomorphism since for each pair $X' \subseteq X$, $Y' \subseteq Y$ in $\mathscr{A}$ with $X/X', Y' \in \mathscr{C}$ there exists a subobject $X'' \subseteq X$ in $\mathrm{fp}(\mathscr{A})$ with $X/X'' \in \mathrm{fp}(\mathscr{C})$ such that $X'' \subseteq X'$. Furthermore, one uses that $\mathrm{Hom}(X'', Y/Y') \cong \varinjlim \mathrm{Hom}(X'', Y/Y'')$ where $Y'' \subseteq Y'$ with $Y'' \in \mathrm{fp}(\mathscr{C})$. Thus the functor f is fully faithful. Finally, f is also dense since any object in $\mathrm{fp}(\mathscr{A}/\mathscr{C})$ is a direct limit of objects in the image of f and therefore a direct summand of some object in the image of f. This completes the proof. □

Having shown that any finite type localizing subcategory of a locally coherent category $\mathscr{A}$ is of the form $\vec{\mathscr{S}}$ for some Serre subcategory of $\mathrm{fp}(\mathscr{A})$ it is natural to ask whether the converse is true. We shall need the following lemma.

Lemma 2.7. *Let $\mathscr{A}$ and $\mathscr{B}$ be a pair of locally coherent categories. Any functor $f: \mathrm{fp}(\mathscr{A}) \to \mathrm{fp}(\mathscr{B})$ extends, up to isomorphism, uniquely to a functor $f^*: \mathscr{A} \to \mathscr{B}$ which commutes with direct limits. This functor has the following properties*:

(1) *f is exact iff f^* is exact.*

(2) *f is right exact iff f^* has a right adjoint $f_*: \mathscr{B} \to \mathscr{A}$. A right adjoint of f^* commutes with direct limits.*

(3) *If f exact, then f is faithful iff f^* is faithful.*

Proof. We shall identify $\mathscr{A} = \mathrm{Lex}(\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab})$ and $\mathscr{B} = \mathrm{Lex}(\mathrm{fp}(\mathscr{B})^{\mathrm{op}}, \mathrm{Ab})$. The functor f induces the functor

$$f_*: (\mathrm{fp}(\mathscr{B})^{\mathrm{op}}, \mathrm{Ab}) \to (\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab}), \qquad X \mapsto X \circ f$$

and this has a left adjoint $f^*: (\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab}) \to (\mathrm{fp}(\mathscr{B})^{\mathrm{op}}, \mathrm{Ab})$ which is determined by the fact that it sends $\mathrm{Hom}(\ ,X)$ to $\mathrm{Hom}(\ ,f(X))$ and preserves coproducts and cokernels. Therefore f^* induces the following commutative diagram where the vertical arrows represent inclusions, since the objects in $\mathrm{Lex}(\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab})$ are precisely the direct limits $\varinjlim \mathrm{Hom}(\ ,X_i)$ of representable functors:

$$\begin{array}{ccc}
\mathrm{fp}(\mathscr{A}) & \xrightarrow{\quad f \quad} & \mathrm{fp}(\mathscr{B}) \\
\downarrow & & \downarrow \\
\mathrm{Lex}(\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab}) & \longrightarrow & \mathrm{Lex}(\mathrm{fp}(\mathscr{B})^{\mathrm{op}}, \mathrm{Ab}) \\
\downarrow & & \downarrow \\
(\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{\quad f^* \quad} & (\mathrm{fp}(\mathscr{B})^{\mathrm{op}}, \mathrm{Ab})
\end{array}$$

(1) Any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ can be written as a direct limit of exact sequences $0 \to X_i \to Y_i \to Z_i \to 0$ in $\mathrm{fp}(\mathscr{A})$. The assertion follows from this fact since f^* commutes with direct limits which are exact.

(2) If f is right exact, then the functor $\mathrm{Lex}(\mathrm{fp}(\mathscr{B})^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Lex}(\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto X \circ f$ is a right adjoint for f^*. Conversely, the existence of a right adjoint implies that f^* is right exact. Thus f is right exact.

(3) Suppose that f is faithful. To show that f^* is faithful assume that $f^*(\varphi) = 0$ for some morphism $\varphi \neq 0$. It follows that $f^*(\mathrm{Im}(\varphi)) = 0$ and therefore $f^*(X/U) = 0$ for some $X \in \mathrm{fp}(\mathscr{A})$ and some proper subobject U. Writing $U = \sum U_i$ as the sum of finitely generated subobjects we obtain $\sum f(U_i) = f^*(U) = f^*(X) = f(X)$ and therefore $f(U_j) = f(X)$ for some j since $f(X)$ is finitely generated. Thus $U_j = X$ since f is faithful. Contradiction. □

Theorem 2.8. *Let $\mathscr{A}$ be a locally coherent category and suppose that $\mathscr{S}$ is a Serre subcategory of* $\mathrm{fp}(\mathscr{A})$*. Then $\vec{\mathscr{S}}$ is a localizing subcategory of finite type of $\mathscr{A}$.*

Proof. The category $\vec{\mathscr{S}}$ is locally coherent since $\mathrm{fp}(\vec{\mathscr{S}}) = \mathscr{S}$ and therefore the inclusion $\vec{\mathscr{S}} \to \mathscr{A}$ admits a right adjoint commuting with direct limits by Lemma 2.7. We conclude from Lemma 2.1 that $\vec{\mathscr{S}}$ is a localizing subcategory of finite type iff $\vec{\mathscr{S}}$ is a Serre subcategory of $\mathscr{A}$. Thus it remains to show that $\vec{\mathscr{S}}$ is a Serre subcategory. To this end consider the quotient functor $q\colon \mathrm{fp}(\mathscr{A}) \to \mathrm{fp}(\mathscr{A})/\mathscr{S}$ which induces an exact functor $q^*\colon \mathscr{A} \to \mathscr{B}$ for $\mathscr{B} = \mathrm{Lex}((\mathrm{fp}(\mathscr{A})/\mathscr{S})^{\mathrm{op}}, \mathrm{Ab})$ by Lemma 2.7. The kernel $\mathrm{Ker}(q^*)$ is a Serre subcategory of $\mathscr{A}$ and we claim that $\vec{\mathscr{S}} = \mathrm{Ker}(q^*)$. Clearly, $\vec{\mathscr{S}} \subseteq \mathrm{Ker}(q^*)$ since q^* commutes with direct limits. To check the converse we use the following criterion. An object $X \in \mathscr{A}$ belongs to $\vec{\mathscr{S}}$ iff any morphism $Y \to X$ with $Y \in \mathrm{fp}(\mathscr{A})$ factors through some object in $\mathscr{S}$ [3, Lemma 4.1]. Now let $\varphi\colon Y \to X$ be such a morphism with $X \in \mathrm{Ker}(q^*)$. Write $X = \varinjlim X_i$ with canonical morphisms $\theta_i\colon X_i \to X$ and $X_i \in \mathrm{fp}(\mathscr{A})$ for all i. We obtain

$$0 = q^*(\varphi) \in \mathrm{Hom}(q^*(Y), q^*(X)) \cong \varinjlim \mathrm{Hom}(q(Y), q(X_i))$$

and it follows directly that there is some j and some morphism $\alpha\colon Y \to X_j$ such that $\varphi = \theta_j \circ \alpha$ and $q(\alpha) = 0$. Thus $\mathrm{Im}(\alpha) \in \mathscr{S}$ and therefore φ factors through some object in $\mathscr{S}$. This finishes the proof. □

Some consequences are as follows.

Corollary 2.9. *Let $f\colon \mathscr{A} \to \mathscr{B}$ be an exact functor between locally coherent categories and suppose that f preserves direct limits and finitely presented objects. If $\mathscr{S} = \mathrm{Ker}(f) \cap \mathrm{fp}(\mathscr{A})$, then $\mathrm{Ker}(f) = \vec{\mathscr{S}}$ and this is a localizing subcategory of finite type.*

Proof. The subcategory $\vec{\mathscr{S}}$ is localizing of finite type by the preceding theorem and the argument used in its proof shows that $\mathrm{Ker}(f) = \vec{\mathscr{S}}$ since $\mathscr{S}$ is the kernel of the restricted functor $\mathrm{fp}(\mathscr{A}) \to \mathrm{fp}(\mathscr{B})$. □

Corollary 2.10. *Let $\mathscr{A}$ be a locally coherent category. There is a bijective correspondence between Serre subcategories of* $\mathrm{fp}(\mathscr{A})$ *and hereditary torsion theories of finite type for $\mathscr{A}$. The correspondence is given by*

$$\mathscr{S} \mapsto (\vec{\mathscr{S}}, \{X \in \mathscr{A} \mid \mathrm{Hom}(\mathscr{S}, X) = 0\}) \quad \textit{and} \quad (\mathscr{T}, \mathscr{F}) \mapsto \mathscr{T} \cap \mathrm{fp}(\mathscr{A}).$$

Proof. Combine Lemma 2.3 and Theorem 2.8. □

Corollary 2.11. *Let $\mathscr{A}$ be a locally coherent category. If $\mathscr{S}$ is a Serre subcategory of* $\mathrm{fp}(\mathscr{A})$*, then the right perpendicular category $\mathscr{S}^{\perp}$ of $\mathscr{S}$ in $\mathscr{A}$ is locally coherent* *and coincides with* $(\vec{\mathscr{S}})^{\perp}$. *In particular the section functor* $\mathscr{A}/\vec{\mathscr{S}} \to \mathscr{A}$ *induces an equivalence between* $\mathscr{A}/\vec{\mathscr{S}}$ *and* $\mathscr{S}^{\perp}$.

Proof. It suffices to show that an object in $\mathscr{A}$ belongs to $\mathscr{S}^{\perp}$ iff it is $\vec{\mathscr{S}}$-closed, since the $\vec{\mathscr{S}}$-closed objects are precisely those of $(\vec{\mathscr{S}})^{\perp}$ by Lemma 2.2. The rest of the assertion then follows from Theorem 2.6. Clearly, any $\vec{\mathscr{S}}$-closed object belongs to $\mathscr{S}^{\perp}$. To verify the converse denote by $q: \mathscr{A} \to \mathscr{A}/\vec{\mathscr{S}}$ the quotient functor and let s be a right adjoint. Let $X \in \mathscr{S}^{\perp}$. It follows immediately that $\mathrm{Hom}(\vec{\mathscr{S}}, X) = 0$ and we obtain therefore the canonical exact sequence $0 \to X \xrightarrow{\xi_X} s \circ q(X) \to C \to 0$ with $C \in \vec{\mathscr{S}}$. We need to show that $C = 0$. Write $C = \varinjlim C_i$ with $C_i \in \mathscr{S}$ for all i. Each canonical morphism $\varphi_i : C_i \to C$ factors through $s \circ q(X)$ since $\mathrm{Ext}^1(\mathscr{S}, X) = 0$, and therefore $\varphi_i = 0$ for all i since $\mathrm{Hom}(\mathscr{S}, s \circ q(X)) = 0$. Thus $C = 0$ and X is $\vec{\mathscr{S}}$-closed. □

We finish this section with an example of a localizing subcategory of finite type. To this end let $\mathscr{A}$ be a locally finitely presented abelian category and let $\mathscr{B} = (\mathrm{fp}(\mathscr{A})^{\mathrm{op}}, \mathrm{Ab})$.

Proposition 2.12. *There exists a finite type localizing subcategory* $\mathscr{C}$ *of* $\mathscr{B}$ *such that the composition of* $\mathscr{A} \to \mathscr{B}$, $X \mapsto \mathrm{Hom}(\ , X)|_{\mathrm{fp}(\mathscr{A})}$ *with the quotient functor* $\mathscr{B} \to \mathscr{B}/\mathscr{C}$ *is an equivalence. The category* $\mathscr{A}$ *is locally coherent if and only if* $\mathscr{B}$ *is locally coherent, equivalently if* $\mathrm{fp}(\mathscr{A})$ *has pseudo-kernels.*

Proof. Recall that $\varphi: X \to Y$ is a *pseudo-kernel* for $\psi: Y \to Z$ if the induced sequence of functors $\mathrm{Hom}(\ , X) \to \mathrm{Hom}(\ , Y) \to \mathrm{Hom}(\ , Z)$ is exact. The first part of the assertion is well known (e.g. [1, Satz 2.7]). The fact that $\mathscr{C}$ is of finite type follows from Lemma 2.4 since the functor $\mathscr{A} \to \mathscr{B}$ commutes with direct limits. If $\mathscr{B}$ is locally coherent, then $\mathscr{A}$ is locally coherent by Theorem 2.6. The converse follows from the well-known fact that for any skeletally small additive category $\mathscr{D}$ the category $\mathrm{fp}(\mathscr{D}^{\mathrm{op}}, \mathrm{Ab})$ is abelian iff $\mathscr{D}$ has pseudo-kernels [1, p. 315]. □

3. The spectrum

Let $\mathscr{A}$ be a Grothendieck category with a generating set $\mathscr{G}$ of objects. Recall that the isomorphism classes of indecomposable injective objects in $\mathscr{A}$ form a set since any indecomposable injective object in $\mathscr{A}$ occurs as the injective envelope of some quotient X/U with $X \in \mathscr{G}$. We denote by $\mathrm{sp}(\mathscr{A})$ the set of isomorphism classes of indecomposable injective objects in $\mathscr{A}$ and call $\mathrm{sp}(\mathscr{A})$ the *spectrum* of $\mathscr{A}$. It will be convenient to identify each isomorphism class in $\mathrm{sp}(\mathscr{A})$ with a representative belonging to it.

Lemma 3.1. *Let* $\mathscr{A}$ *be a locally finitely presented Grothendieck category and let* $U = \prod_{M \in \mathrm{sp}(\mathscr{A})} M$. *Any object* $X \in \mathscr{A}$ *embeds in some product of copies of* U.

Proof. First observe that $\operatorname{Hom}(X,U) \neq 0$ for every finitely generated object $X \neq 0$ since there exists a maximal subobject $Y \subseteq X$ so that an injective envelope $X/Y \to M$ provides a non-zero morphism $X \to M$ with $M \in \operatorname{sp}(\mathscr{A})$. Given an arbitrary object X, let $\delta: X \to \prod_{\varphi \in \operatorname{Hom}(X,U)} U$ be the map defined by $\delta_\varphi = \varphi$ for all $\varphi \in \operatorname{Hom}(X,U)$. We claim that $\operatorname{Ker}(\delta) = 0$. To show this assume that $\operatorname{Ker}(\delta) \neq 0$ and choose a finitely generated non-zero object $X' \subseteq \operatorname{Ker}(\delta)$. There is a non-zero morphism $X' \to U$ which extends to a non-zero morphism $\varphi: X \to U$. But φ factors through δ and therefore $\varphi(X') = 0$. This contradiction finishes the proof. □

Let $\mathscr{A}$ be a Grothendieck category and $\mathscr{C}$ a localizing subcategory of $\mathscr{A}$. It is well known that $\mathscr{C}$ and $\mathscr{A}/\mathscr{C}$ are also Grothendieck categories [3, III.4, Proposition 9]. Denote by s the section functor $\mathscr{A}/\mathscr{C} \to \mathscr{A}$ and for each object $X \in \mathscr{A}$ let $E(X)$ be its injective envelope in $\mathscr{A}$. The assignment $X \mapsto E(X)$ induces an injective map $\operatorname{sp}(\mathscr{C}) \to \operatorname{sp}(\mathscr{A})$ and $X \mapsto s(X)$ induces an injective map $\operatorname{sp}(\mathscr{A}/\mathscr{C}) \to \operatorname{sp}(\mathscr{A})$. We consider both maps as identification. They satisfy $\operatorname{sp}(\mathscr{A}) = \operatorname{sp}(\mathscr{C}) \cup \operatorname{sp}(\mathscr{A}/\mathscr{C})$ and $\operatorname{sp}(\mathscr{C}) \cap \operatorname{sp}(\mathscr{A}/\mathscr{C}) = \emptyset$ [3, III.3, Corollaire 2].

Proposition 3.2. *Let $\mathscr{A}$ be a Grothendieck category and $(\mathscr{T},\mathscr{F})$ be a hereditary torsion theory for $\mathscr{A}$. If $\mathscr{A}/\mathscr{T}$ is locally finitely presented, then $\mathscr{U} = \mathscr{F} \cap \operatorname{sp}(\mathscr{A}) = \operatorname{sp}(\mathscr{A}/\mathscr{C})$ cogenerates $(\mathscr{T},\mathscr{F})$ in the following sense:*

(1) *$X \in \mathscr{F}$ iff X embeds in some product of copies of $U = \prod_{M \in \mathscr{U}} M$.*

(2) *$X \in \mathscr{T}$ iff $\operatorname{Hom}(X,\mathscr{U}) = 0$.*

Proof. (1) We use the fact that the left exact functor s induces an equivalence between the full subcategory of injective objects $\operatorname{inj}(\mathscr{A}/\mathscr{C})$ and $\operatorname{inj}(\mathscr{A}) \cap \mathscr{F}$. Given $X \in \mathscr{F}$ let $X \to M$ be an injective envelope. It is clear that $M \in \mathscr{F}$. Using Lemma 3.1 we find a monomorphism $q(M) \to \prod_{i\in I} q(U)$ for some set I. The composition $X \to M \to \prod_{i \in I} U$ gives the desired monomorphism. For the converse use that $\mathscr{F}$ is closed under products and subobjects.

(2) Given $X \in \mathscr{T}$, clearly $\operatorname{Hom}(X,\mathscr{U}) = 0$. For the converse suppose that $X \notin \mathscr{T}$. By definition there is a non-zero morphism $\varphi: X \to Y$ for some $Y \in \mathscr{F}$. Using part (1) it follows that $\operatorname{Hom}(X,\mathscr{U}) \neq 0$ and the proof is finished. □

4. A topology for the spectrum

In [7] Ziegler introduces a topology on the isomorphism classes of indecomposable pure injective Λ-modules for a ring Λ. We extend this concept of a topology on $\operatorname{sp}(\operatorname{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ to the spectrum of an arbitrary locally coherent category. Note that this is independent from all previous results in this paper.

Let $\mathscr{A}$ be a locally coherent category. For a subset $\mathscr{U}$ of $\operatorname{sp}(\mathscr{A})$ denote by $\Sigma(\mathscr{U})$ the Serre subcategory of $\operatorname{fp}(\mathscr{A})$ formed by the objects $X \in \operatorname{fp}(\mathscr{A})$ satisfying $\operatorname{Hom}(X,\mathscr{U}) = 0$. For a subcategory $\mathscr{S}$ of $\operatorname{fp}(\mathscr{A})$ let $\Upsilon(\mathscr{S}) = \{M \in \operatorname{sp}(\mathscr{A}) \mid \operatorname{Hom}(\mathscr{S},M) = 0\}$.

Lemma 4.1. *The assignment*

$$\mathscr{U} \mapsto \overline{\mathscr{U}} = \Upsilon \circ \Sigma(\mathscr{U})$$

is a closure operator on the spectrum $\mathrm{sp}(\mathscr{A})$ *of* $\mathscr{A}$, *i.e. the subsets* $\mathscr{U} \subseteq \mathrm{sp}(\mathscr{A})$ *satisfying* $\overline{\mathscr{U}} = \mathscr{U}$ *form the closed sets of a topology on* $\mathrm{sp}(\mathscr{A})$.

Proof. Following Kuratowski's axiomatization of a topological space we need to verify that

(i) $\overline{\emptyset} = \emptyset$;
(ii) $\mathscr{U} \subseteq \overline{\mathscr{U}}$ for every subset $\mathscr{U}$;
(iii) $\overline{\overline{\mathscr{U}}} = \overline{\mathscr{U}}$ for every subset $\mathscr{U}$;
(iv) $\overline{\mathscr{U}_1 \cup \mathscr{U}_2} = \overline{\mathscr{U}_1} \cup \overline{\mathscr{U}_2}$ for every pair of subsets $\mathscr{U}_1$ and $\mathscr{U}_2$.

The conditions (i)–(iii) are easily checked and it remains to show (iv). From $\Sigma(\mathscr{U}_1 \cup \mathscr{U}_2) \subseteq \Sigma(\mathscr{U}_1) \cap \Sigma(\mathscr{U}_2)$ it follows that $\overline{\mathscr{U}_1} \cup \overline{\mathscr{U}_2} \subseteq \overline{\mathscr{U}_1 \cup \mathscr{U}_2}$. Now choose $M \in \mathrm{sp}(\mathscr{A})$ such that $M \notin \overline{\mathscr{U}_1} \cup \overline{\mathscr{U}_2}$. We claim that this implies $M \notin \overline{\mathscr{U}_1 \cup \mathscr{U}_2}$. From the definitions we may choose non-zero morphisms $\varphi_i : X_i \to M$ such that $X_i \in \Sigma(\mathscr{U}_i)$. We have $\mathrm{Im}(\varphi_1) \cap \mathrm{Im}(\varphi_2) \neq 0$ since M is indecomposable injective. Choosing $U \subseteq \mathrm{Im}(\varphi_1) \cap \mathrm{Im}(\varphi_2)$ finitely generated one uses Lemma 1.1 to find finitely generated subobjects $Y_i \subseteq X_i$ such that $\varphi_i(Y_i) = U$. We obtain the following exact commutative diagram where the vertical morphisms are the canonical monomorphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \xrightarrow{\left[\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right]} & Y_1 \coprod Y_2 & \longrightarrow & U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & X_1 \coprod X_2 & \xrightarrow{[\varphi_1\ \varphi_2]} & M & &
\end{array}$$

The morphisms ψ_i being epimorphisms we find finitely generated subobjects W_i of W such that $\psi_i(W_i) = Y_i$. Let $X = Y_1 \coprod Y_2 / \left[\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right](W_1 + W_2)$. We have $X \in \mathrm{fp}(\mathscr{A})$ since $\mathrm{fp}(\mathscr{A})$ is abelian and it is easily checked that $\mathrm{Hom}(X, M) \neq 0$. On the other hand, $X \in \Sigma(\mathscr{U}_1 \cup \mathscr{U}_2)$ since X is a quotient of each Y_i. Therefore $M \notin \overline{\mathscr{U}_1 \cup \mathscr{U}_2}$ and the proof is complete. □

It has been observed by I. Herzog and M. Prest that Ziegler's closed sets in $\mathrm{sp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ are in bijective correspondence to Serre subcategories of $\mathrm{fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$. In our context this observation takes the following form.

Theorem 4.2. *Let* $\mathscr{A}$ *be a locally coherent category. There is a bijective inclusion reversing correspondence between Serre subcategories of* $\mathrm{fp}(\mathscr{A})$ *and closed subsets of* $\mathrm{sp}(\mathscr{A})$. *The correspondence is given by*

$$\mathscr{S} \mapsto \Upsilon(\mathscr{S}) \quad \text{and} \quad \mathscr{U} \mapsto \Sigma(\mathscr{U}).$$

Proof. We check that the assignments are inverse to each other. Given a Serre subcategory $\mathscr{S}$ of $\mathrm{fp}(\mathscr{A})$, the pair $(\mathscr{T}, \mathscr{F})$ with $\mathscr{T} = \vec{\mathscr{S}}$ and $\mathscr{F} = \{X \mid \mathrm{Hom}(\mathscr{S}, X) = 0\}$

forms a hereditary torsion theory of finite type by Corollary 2.10. Thus $\mathscr{A}/\mathscr{T}$ is locally finitely presented by Theorem 2.6 and it follows from Proposition 3.2 that $\Sigma \circ \Upsilon(\mathscr{S}) = \mathscr{S}$ since $\Upsilon(\mathscr{S}) = \mathscr{F} \cap \mathrm{sp}(\mathscr{A})$. In particular this shows that $\Upsilon(\mathscr{S})$ is closed. Conversely, $\Upsilon \circ \Sigma(\mathscr{U}) = \mathscr{U}$ is clear since $\mathscr{U}$ is closed. □

Corollary 4.3. *Let $\mathscr{A}$ be a locally coherent category. There is a bijective correspondence between closed subsets of* $\mathrm{sp}(\mathscr{A})$ *and hereditary torsion theories of finite type for $\mathscr{A}$ given by*

$$\mathscr{U} \mapsto (\{X \mid \mathrm{Hom}(X, \mathscr{U}) = 0\}, \{X \mid X \subseteq \textstyle\prod_I (\prod_{M \in \mathscr{U}} M) \textit{ for some } I\})$$

and

$$(\mathscr{T}, \mathscr{F}) \mapsto \mathrm{sp}(\mathscr{A}) \cap \mathscr{F}.$$

Proof. Combine Corollary 2.10 and Theorem 4.2. □

Let $\mathscr{C}$ be a finite type localizing subcategory of $\mathscr{A}$. Then it is shown in Theorem 2.6 that $\mathscr{C}$ and $\mathscr{A}/\mathscr{C}$ are locally coherent and it is therefore natural to ask how the topologies of $\mathrm{sp}(\mathscr{C})$ and $\mathrm{sp}(\mathscr{A}/\mathscr{C})$ are related to that of $\mathrm{sp}(\mathscr{A})$.

Corollary 4.4. *The topologies of* $\mathrm{sp}(\mathscr{C})$ *and* $\mathrm{sp}(\mathscr{A}/\mathscr{C})$ *coincide with their topologies induced from* $\mathrm{sp}(\mathscr{A})$.

Proof. Viewing $\mathrm{sp}(\mathscr{C})$ and $\mathrm{sp}(\mathscr{A}/\mathscr{C})$ in the natural way as subsets of $\mathrm{sp}(\mathscr{A})$ the assertion follows from Theorem 4.2. □

Recall that a topological space $\mathscr{X}$ is *quasi-compact* provided that for every family $(\mathscr{U}_i)_{i \in I}$ of open subsets $\mathscr{X} = \bigcup_{i \in I} \mathscr{U}_i$ implies $\mathscr{X} = \bigcup_{i \in J} \mathscr{U}_i$ for some finite subset J of I. A subset of $\mathscr{X}$ is quasi-compact if it is quasi-compact with respect to the induced topology.

Rephrasing Theorem 4.2 the assignment $\mathscr{S} \mapsto \{M \in \mathrm{sp}(\mathscr{A}) \mid \mathrm{Hom}(\mathscr{S}, M) \neq 0\}$ gives a bijective correspondence between Serre subcategories of $\mathrm{fp}(\mathscr{A})$ and open subsets of $\mathrm{sp}(\mathscr{A})$. Note that for every subcategory $\mathscr{C}$ of $\mathrm{fp}(\mathscr{A})$ one has $\{M \in \mathrm{sp}(\mathscr{A}) \mid \mathrm{Hom}(\mathscr{C}, M) \neq 0\} = \{M \in \mathrm{sp}(\mathscr{A}) \mid \mathrm{Hom}(\mathscr{S}, M) \neq 0\}$ where $\mathscr{S}$ denotes the smallest Serre subcategory of $\mathrm{fp}(\mathscr{A})$ containing $\mathscr{C}$. This has the following consequence.

Corollary 4.5. *Any open subset $\mathscr{U}$ of* $\mathrm{sp}(\mathscr{A})$ *is of the form* $\{M \in \mathrm{sp}(\mathscr{A}) \mid \mathrm{Hom}(\mathscr{S}, M) \neq 0\}$ *for some Serre subcategory $\mathscr{S}$ of* $\mathrm{fp}(\mathscr{A})$. *For such an open set $\mathscr{U}$ the following are equivalent:*

(1) *$\mathscr{U}$ is quasi-compact.*

(2) $\mathscr{U} = \{M \in \mathrm{sp}(\mathscr{A}) \mid \mathrm{Hom}(X, M) \neq 0\}$ *for some $X \in \mathscr{S}$.*

(3) *$\mathscr{S}$ is the smallest Serre subcategory containing X for some $X \in \mathscr{S}$.*

Writing $(X) = \{M \in \text{sp}(\mathscr{A}) \mid \text{Hom}(X,M) \neq 0\}$ for each $X \in \text{fp}(\mathscr{A})$ we obtain a basis of open sets as follows.

Corollary 4.6. *The family of all subsets of the form (X) for some $X \in \text{fp}(\mathscr{A})$ forms a basis of quasi-compact open sets for the topology on* $\text{sp}(\mathscr{A})$.

We conclude this paper with two examples. Let $\mathscr{C}$ be a skeletally small additive category and suppose that the category $(\mathscr{C}, \text{Ab})$ of additive functors from $\mathscr{C}$ to Ab is locally coherent. The following proposition is an immediate consequence of the preceding result and Yoneda's lemma.

Proposition 4.7. *The spectrum of $(\mathscr{C}, \text{Ab})$ is quasi-compact if and only if there exists an object $X \in \mathscr{C}$ such that $M(X) \neq 0$ for all $M \in \text{sp}(\mathscr{C}, \text{Ab})$.*

Let $\mathscr{A}$ be a skeletally small abelian category. The Serre subcategories of $\mathscr{A}$ form a small partially ordered set which we denote by $\mathscr{L}(\mathscr{A})$. If $(\mathscr{C}_i)_{i \in I}$ is a family in $\mathscr{L}(\mathscr{A})$, then their intersection $\bigcap_{i \in I} \mathscr{C}_i$ is again a Serre subcategory and one obtains $\bigcup_{i \in I} \mathscr{C}_i$ as the intersection of all $\mathscr{C} \in \mathscr{L}(\mathscr{A})$ with $\mathscr{C}_i \subseteq \mathscr{C}$ for all $i \in I$. Now observe that $\text{Lex}(\mathscr{A}^{\text{op}}, \text{Ab})$ is a locally coherent category. It follows from Theorem 4.2 that the assignment

$$\mathscr{C} \mapsto \{M \in \text{sp}(\text{Lex}(\mathscr{A}^{\text{op}}, \text{Ab})) \mid M(\mathscr{C}) \neq 0\}$$

gives a lattice isomorphism between $\mathscr{L}(\mathscr{A})$ and the lattice of open subsets of $\text{sp}(\text{Lex}(\mathscr{A}^{\text{op}}, \text{Ab}))$.

Proposition 4.8. *$\mathscr{L}(\mathscr{A})$ is isomorphic to the lattice of open subsets of some topological space.*

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